Problem with a solution proposed by Arkady Alt , San Jose , California, USA

Let $m,n \geq 2$ be positive integer numbers such that $m \geq n$. Prove that for any positive real a and b holds inequality

$$\left(\frac{a^m + b^m}{a^{m-1} + b^{m-1}}\right)^{n+1} \geq \frac{a^{n+1} + b^{n+1}}{2}.$$

Solution.

First we will prove the inequality for $m = n$, namely

(**1**) $$\left(\frac{a^n + b^n}{a^{n-1} + b^{n-1}}\right)^{n+1} \geq \frac{a^{n+1} + b^{n+1}}{2}, n \geq 2.$$

Proof of (**1**). (**By Math Induction**).

1. *Base of Math. Induction.*

For $n = 2$ we have

$$\left(\frac{a^2 + b^2}{a + b}\right)^3 \geq \frac{a^3 + b^3}{2} \iff 2(a^2 + b^2)^3 \geq (a + b)^3(a^3 + b^3)$$ and

$$2(a^2 + b^2)^3 - (a + b)^3(a^3 + b^3) = (ab + a^2 + b^2)(a - b)^4 \geq 0.$$

2. *Step of Math Induction.*

Let $a_n := \left(\frac{a^n + b^n}{a^{n-1} + b^{n-1}}\right)^{n+1}$ and $b_n := \frac{a^{n+1} + b^{n+1}}{2}$.

Then $$\frac{a_{n+1}}{a_n} \geq \frac{b_{n+1}}{b_n} \iff \frac{(a^{n+1} + b^{n+1})^{n+2}(a^{n-1} + b^{n-1})^{n+1}}{(a^n + b^n)^{n+2}(a^n + b^n)^{n+1}} \geq \frac{a^{n+2} + b^{n+2}}{a^{n+1} + b^{n+1}} \iff$$

$$\frac{(a^{2n} + a^{n+1}b^{n-1} + a^{n-1}b^{n+1} + b^{2n})^{n+1}}{(a^{2n} + 2a^n b^n + b^{2n})^{n+1}} \geq \frac{a^{2n+2} + a^{n+2}b^n + a^n b^{n+2} + b^{n+2}}{a^{2n+2} + 2a^{n+1}b^{n+1} + b^{2n+2}} \iff$$

$$\left(1 + \frac{a^{n+1} + a^{n-1} - 2a^n b^n}{a^{2n} + 2a^n b^n + b^{2n}}\right)^{n+1} \geq 1 + \frac{a^{n+2}b^n - 2a^{n+1}b^{n+1} + a^n b^{n+2}}{a^{2n+2} + 2a^{n+1}b^{n+1} + b^{2n+2}} \iff$$

$$\left(1 + \frac{a^{n-1}b^{n-1}(a - b)^2}{(a^n + b^n)^2}\right)^{n+1} \geq 1 + \frac{a^n b^n (a - b)^2}{(a^{n+1} + b^{n+1})^2}$$

and by Bernoulli Inequality we have

$$\left(1 + \frac{a^{n-1}b^{n-1}(a - b)^2}{(a^n + b^n)^2}\right)^{n+1} \geq 1 + \frac{(n + 1)a^{n-1}b^{n-1}(a - b)^2}{(a^n + b^n)^2}.$$

So, suffice to prove $\frac{(n + 1)a^{n-1}b^{n-1}(a - b)^2}{(a^n + b^n)^2} \geq \frac{a^n b^n (a - b)^2}{(a^{n+1} + b^{n+1})^2}$.

Since $n \geq 2$ and $a^{n+1} + b^{n+1} \geq \frac{(a^n + b^n)(a + b)}{2}$ then

$$\frac{(n + 1)a^{n-1}b^{n-1}(a - b)^2}{(a^n + b^n)^2} - \frac{a^n b^n (a - b)^2}{(a^{n+1} + b^{n+1})^2} \geq \frac{3a^{n-1}b^{n-1}(a - b)^2}{(a^n + b^n)^2} - \frac{4a^n b^n (a - b)^2}{(a + b)^2(a^n + b^n)^2} =$$

$$\frac{a^{n-1}b^{n-1}(a - b)^2}{(a + b)^2(a^n + b^n)^2}\left(3(a + b)^2 - 4ab\right) = \frac{a^{n-1}b^{n-1}(a - b)^2}{(a + b)^2(a^n + b^n)^2}(3a^2 + 2ab + 3b^2) \geq 0).$$

Now note, that for any $m \geq n \geq 1$ holds inequality

(**2**) $$\left(\frac{a^m + b^m}{a^{m-1} + b^{m-1}}\right)^n \geq \frac{a^n + b^n}{2}.$$

Indeed, since $\frac{a^m + b^m}{a^{m-1} + b^{m-1}}$ is increasing in $m \in \mathbb{N}$ ($\frac{a^{m+1} + b^{m+1}}{a^m + b^m} \geq \frac{a^m + b^m}{a^{m-1} + b^{m-1}} \iff$ $(a^{m+1} + b^{m+1})(a^{m-1} + b^{m-1}) \geq (a^m + b^m)^2 \iff (a - b)^2 \geq 0$) then

$m \geq n \Rightarrow \dfrac{a^m + b^m}{a^{m-1} + b^{m-1}} \geq \dfrac{a^n + b^n}{a^{n-1} + b^{n-1}} \Rightarrow \left(\dfrac{a^m + b^m}{a^{m-1} + b^{m-1}}\right)^n \geq \left(\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}}\right)^n$

and also we have $\left(\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}}\right)^n \geq \dfrac{a^n + b^n}{2} \Leftrightarrow \sqrt[n]{\dfrac{a^n + b^n}{2}} \geq \sqrt[n-1]{\dfrac{a^{n-1} + b^{n-1}}{2}}$.

Hence $\left(\dfrac{a^m + b^m}{a^{m-1} + b^{m-1}}\right)^n \geq \dfrac{a^n + b^n}{2}$.

Since $\left(\dfrac{a^n + b^n}{a^{n-1} + b^{n-1}}\right)^{n+1} \geq \dfrac{a^{n+1} + b^{n+1}}{2}, n \geq 2$ *and* $\left(\dfrac{a^m + b^m}{a^{m-1} + b^{m-1}}\right)^n \geq \dfrac{a^n + b^n}{2}, m \geq n$

then, inequality $\boxed{\left(\dfrac{a^m + b^m}{a^{m-1} + b^{m-1}}\right)^{n+1} \geq \dfrac{a^{n+1} + b^{n+1}}{2}}$ *holds for any* $m \geq n \geq 2$.

Remark.

Note that in the case $n = 1$ inequality (**1**) becomes $\left(\dfrac{a + b}{2}\right)^2 \geq \dfrac{a^2 + b^2}{2}$

which isn't right.